Theoretical Computer Science 108 (1993) 25–43
Elsevier

Möbius functions and confluent semi-commutations*

Volker Diekert

Institut für Informatik, Technische Universität München, Arcisstr. 21, D-8000 München 2, Germany

Abstract

Diekert, V., Möbius functions and confluent semi-commutations, Theoretical Computer Science 108 (1993) 25–43.

Diekert (1988) settled a well-known conjecture on unambiguous liftings to words of Möbius functions for free partially commutative monoids. He established thereby a new bridge from the theory of formal power series to the theory of complete semi-Thue systems. As an open question, it was asked whether this approach has a generalization to a relative situation where we do not lift to the free (word-) case but only to some level of less commutation. This question has been reconsidered by König (1992) and a conjecture has been formulated there. Our results solve the open problem and settle this conjecture (Remark 4.8).

We show that there is a canonical one-to-one correspondence between unambiguous Möbius functions and confluent semi-commutation systems (Theorem 5.1). This identification is due to some graph-theoretic characterization obtained by Diekert et al. (1991) and which allows one to apply some interesting complexity results (Section 6).

Our results can be viewed as a contribution to the combinatorial theory of Möbius functions and to the theory of rewriting on traces.

1. Introduction

Free partially commutative monoids have been introduced in combinatorics by Cartier and Foata in [2]. In computer science they are used as an algebraic model for concurrency. This is mainly due to the work of Mazurkiewicz [14]. Let us refer to [1, 7, 15, 20] for more background information.

From the beginning of the theory, Möbius functions have been of interest. The basic formula is due to Cartier and Foata [2] stating that the Möbius function can be expressed as a polynomial over the cliques of the underlying independence graph.

Correspondence to: V. Diekert, Institut für Informatik der Universität Stuttgart, Breitwiesenstr. 20–22, D-7000 Stuttgart 80, Germany. Email: diekert@informatik.uni-stuttgart.de.

* Research partially supported by a grant of the Deutsche Forschungsgemeinschaft and by the ESPRIT Basic Research Action 3166 ASMICS.

(This is a proper generalization of the classical Möbius inversion from number theory.) Choffrut reported in [3] a question of Perrin whether any Möbius function can be lifted to a polynomial over words such that the formal inverse of the lifting is a characteristic series over a rational cross section. This allows convenient calculations with representing words instead of traces and in the positive case it is said that the Möbius function can be lifted unambiguously to words. The answer in [6] showed that there is such a lifting if and only if there exists a presentation of the free partially commutative monoid by some complete semi-Thue system. (For a homological interpretation of this result see [11].) The existence of such a complete presentation is characterized by the fact that the independence alphabet is a comparability graph, [19]. Hence, by Golumbic [9] the question whether the Möbius function can be lifted unambiguously to words is decidable in polynomial time.

In the present work, we consider the general situation of two free partially commutative monoids M and M' defined by independence alphabets (X, I) and (X, I'), respectively, where $I \subseteq I'$. A lifting of the Möbius function of M' to a polynomial over M is called *unambiguous* if its formal inverse is the characteristic series over a rational cross section of M' in M. Of course, this generalizes the liftings to the word-case, where we have $I = \emptyset$. Our main motivation to study this situation, however, results from the interest in semi-commutation systems. These systems were introduced by Clerbout in [4] and they received much attention recently, see e.g. [13]. They also describe some phenomena of concurrency; in particular, they are of interest as a semantics for Petri nets, see [18]. From another viewpoint, a semi-commutation system can be interpreted as a certain trace replacement system over some free partially commutative monoid. In this interpretation a semi-commutation system is always noetherian and an important question is whether it is confluent. We will show here that there is a canonical one-to-one correspondence between unambiguous liftings of Möbius functions and confluent semi-commutations. This correspondence associates with a confluent semi-commutation system the unambiguous lifting of the Möbius function which is obtained by the formal inverse of the characteristic power series over the irreducible traces. Thus, the critical-pair-property of confluence has a translation into the combinatorics on Möbius functions.

We obtain our result in three steps. First, we show that an unambiguous lifting defines in a natural way a confluent semi-commutation system. Then we establish a mapping in the opposite direction and finally we show that these mappings are bijective. This is the most difficult part. The technical calculations use a graph-theoretic characterization for the confluence of semi-commutation systems, which can be found in [8]. This allows us additionally to prove some interesting complexity results. For example, a natural question in our context is whether there exists an unambiguous lifting of the Möbius function between two given free partially commutative monoids. It turns out that this question is complete for the second level of the polynomial-time hierarchy. Thus, it is (to our present knowledge) much more difficult than the tractable case where we ask for unambiguously liftings to words, only. This looks as a jump from polynomial-time to the second level of the hierarchy.

However, the intermediate NP-completeness (Co-NP-completeness) occurs if the number of commutation rules between the two monoids differs only by some fixed constant.

2. Preliminaries and basic notations

In what follows X denotes a finite alphabet. (The extension to infinite alphabets with the necessary modifications is straightforward and left to the reader.) An *independence relation* of X is an irreflexive symmetric relation $I \subseteq X \times X$ and the complement $D = X \times X \setminus I$ is called the *dependence relation*. The pair (X, I) is called an *independence alphabet*. A *free partially commutative monoid* is a quotient monoid $X^*/\{ab = ba \mid (a, b) \in I\}$ for X and $I \subseteq X \times X$ as above. It is denoted by $M(X, I)$ or simply by M if the reference to (X, I) is clear. According to Mazurkiewicz [14] an element of $M(X, I)$ is called a *trace* and $M(X, I)$ is also called a *trace monoid*.

It is well known and used throughout here that every trace can be identified with its *dependence graph*. This is a labelled acyclic graph, defined in the following way:

Let $t = [a_1 \ldots a_n] \in M$ be a congruence class of the string $a_1 \ldots a_n \in X^*$, with $n \geqslant 0$, $a_i \in X$ for $1 \leqslant i \leqslant n$. Then take n vertices $v_1, \ldots, v_n$, label vertex v_i with a_i for $1 \leqslant i \leqslant n$ and introduce a directed arc from v_i to v_j if and only if $i < j$ and $(a_i, a_j) \in D$.

Taking the transitive closure, we can view a trace t also as a labelled partial order. Thus, if we think of a string as a total order then the associated trace is the restriction to the partial order where we remember the ordering between dependent letters, only. In fact, if (X, I) is known, then we may omit redundant arcs in the graphical representation of a trace and we may draw its Hasse diagram, only. Thus, if a trace contains an arc from x to y and an arc from y to z, then (usually) no arc between x and z shown.

The ring $Z\langle\langle M \rangle\rangle$ of *formal power series* over M is the set of mappings from X to the integers Z. It has a ring structure by the addition

$$(f+g)(t) = f(t) + g(t)$$

and by the product

$$(fg)(t) = \sum_{t_1 t_2 = t} f(t_1) g(t_2),$$

for $f, g: X \to Z$ and $t \in M$.

Power series are also written as formal sums

$$f = \sum_{t \in M} f(t) t,$$

which is justified by the natural embedding $M \subsetneq Z\langle\langle M \rangle\rangle$ which identifies the element $t \in M$ with the characteristic function $\chi_{\{t\}}: M \to Z$.

A *formal inverse* of a power series $f \in Z\langle\langle M \rangle\rangle$ is an element $g = f^{-1} \in Z\langle\langle M \rangle\rangle$ such that $f \cdot g = 1$. According to the embedding mentioned above, 1 denotes the neutral element of M and the multiplicative unit of $Z\langle\langle M\rangle\rangle$, which maps the empty trace to the integer one and all other traces to zero.

It is clear that $f\in Z\langle\langle M\rangle\rangle$ has a formal inverse if and only if the constant term is a unit of Z, i.e., $f(1)=\pm 1$. In this case we can compute the formal inverse inductively by

$$f^{-1}(1)=f(1),$$

$$f(1)f^{-1}(t)=-\sum_{\substack{t_1t_2=t\\ t_1\neq 1}} f(t_1)f^{-1}(t_2) \quad \text{for } t\neq 1.$$

The *support* of a power series $f\in Z\langle\langle M\rangle\rangle$ is the set $supp(f)=\{t\in M \mid f(t)\neq 0\}$. A power series with finite support is called a *polynomial.*

An *independence clique* (or simply a *clique*) is a finite subset $F\subseteq X$ such that $(a,b)\in I$ for all $a,b\in F$, $a\neq b$. (Elsewhere, e.g. [6], a clique is also called a *step.*) A clique F yields a well-defined trace $[F]$ by taking the product over its elements: $[F]=\prod_{a\in F}a$. The set of all cliques of M is denoted by $\mathscr{F}$ or by $\mathscr{F}(X,I)$. A fundamental result of Cartier and Foata [2, Theorem 1.2] says that the polynomial

$$\mu_M=\sum_{F\in\mathscr{F}}(-1)^{|F|}[F]$$

is the *Möbius function* of the monoid M. This means that the formal inverse of the function μ_M is the constant function with value 1. (Surprisingly, we will never use the result of Cartier and Foata. It turns out to be a special case of our claculations that the constant function with value 1 is the formal inverse of the polynomial μ_M defined by the formula above.)

A *trace replacement system* over M is a subset $S\subseteq M\times M$. The elements of S are called rules. They are written in the form $l\Rightarrow r$ $(l\Leftrightarrow r)$ for $(l,r)\in S$ (for $(l,r),(r,l)\in S$). A system S defines a relation $\underset{S}{\Rightarrow}$ by $s\underset{S}{\Rightarrow}t$ if $s=ulv$, $t=urv$ for some $u,v\in M$ and $(l,r)\in S$. We denote by $\overset{*}{\underset{S}{\Rightarrow}}$, the reflexive and transitive closure and by $\overset{*}{\underset{S}{\Leftrightarrow}}$, the symmetric, reflexive, and transitive closure of $\underset{S}{\Rightarrow}$. The relation $\overset{*}{\underset{S}{\Leftrightarrow}}$ is a congruence of M and the quotient monoid by this congruence is denoted by M/S. The system S is called *noetherian* if there are no infinite derivation chains $t_0\underset{S}{\Rightarrow}t_1\underset{S}{\Rightarrow}t_2\underset{S}{\Rightarrow}\cdots$, and *confluent* if or all $t_1\overset{*}{\underset{S}{\Leftarrow}}t\overset{*}{\underset{S}{\Rightarrow}}t_2$ there exists some $\hat{t}\in M$ such that $t_1\overset{*}{\underset{S}{\Rightarrow}}\hat{t}\overset{*}{\underset{S}{\Leftarrow}}t_2$.

If the condition for confluence is asked only for all $t_1\underset{S}{\Leftarrow}t\underset{S}{\Rightarrow}t_2$ then S is called *locally confluent*. It is a classical result that for noetherian systems local confluence is equivalent to confluence, [17]. A system which is noetherian and (locally) confluent is called *complete*. For complete systems the set of irreducible traces, $Irr(S)=\{t\in M \mid$ for no $s\in M$ it holds that $t\underset{S}{\Rightarrow}s\}$, is in canonical bijection with the quotient M/S.

The main interest in complete systems is due to the fact that they provide us with an effective procedure for solving word problems. This is done by computing irreducible normal forms. Note that the use of traces might be essential: There are quotient monoids of X^* which do not have any presentation by a finite complete semi-Thue system $S \subseteq X^* \times X^*$ but which are presentable by some finite complete trace replacement system over $M(X, I)$ for some suitable independence relation I. (In fact, trace monoids themselves yield such examples.)

The replacement systems which are considered here are very simple. We are nearly exclusively concerned with *semi-commutation rules* $ab \Rightarrow ba$ for some dependent letters $a, b \in X$. They are defined with respect to some *semi-independence alphabet* which is an irreflexive subset $SI \subseteq X \times X$. The associated *semi-commutation system* SC is the set of rules $SC = C \cup S$, where $C = \{ab \Leftrightarrow ba \mid (a,b), (b,a) \in SI\}$ is the set of symmetric (commutation-) rules and $S = \{ab \Rightarrow ba \mid (a,b) \in SI,\ (b,a) \notin SI\}$ is the set of asymmetric rules. Furthermore, we associate with SI the following two independence alphabets and trace monoids[1]: $(X, I) = (X, SI \cap SI^{-1})$, $(X, I') = (X, SI \cup SI^{-1})$, $M = M(X, I)$, and $M' = M(X, I')$. It is clear that we have $M = X^*/C$ and $M' = X^*/SC = M/S$. Here we view S as a trace replacement system $S \subseteq M \times M$ and a simple observation shows that it is noetherian. It follows that we can test confluence by local confluence; hence, the semi-commutation system $SC \subseteq X^* \times X^*$ is confluent if and only if $S \subseteq M \times M$ is complete.

The restriction to semi-commutations in our context is also due to the following fact:

Remark 2.1 (Diekert et al. [8, Theorem 2.3]). Let $M = M(X, I)$, $M' = M(X, I')$ be trace monoids such that $I \subseteq I'$, and let $R \subseteq M \times M$ be any noetherian trace replacement system such that $M/R = M'$. Then R is complete if and only if the semi-commutation system $SC = \{ab \Rightarrow ba \mid a, b \in X, ab \Rightarrow ba \in R\}$ is confluent.

Another result from [8] (where the "only-if" part is crucial below) is Ochmanski's criterion for the confluence of a semi-commutation system. (This result has been independently obtained in [21].) The best formulation for this criterion takes the dependence relation into account. For a semi-independence relation $SI \subseteq X \times X$ we define the *semi-dependence alphabet* by (X, SD), where $SD = X \times X \setminus SI$ is the complement of SI. We identify (X, SD) with a graph where X is the set of vertices. The set of edges is divided into a set of undirected edges and directed arcs. For $a, b \in X$ an undirected edge between a and b means $(a,b), (b,a) \in SD$ whereas a directed arc from a to b means $(a,b) \in SD$, $(b,a) \notin SD$. (Here and in what follows the notation *edge* refers to *undirected* whereas *arc* refers to *directed*.)

A directed cycle in (X, SD) is a sequence $(x_1, \ldots, x_n)$ such that $(x_i, x_{i+1}) \in SD$ for all $i \bmod n$. An undirected chord of a cycle $(x_1, \ldots, x_n)$ is a pair $(x_i, x_j) \in SD \cup SD^{-1}$ such that $2 \leqslant |j - i| \leqslant n - 2$.

Remark 2.2 (Diekert et al. [8, Theorem 2.1]). Let $SI \subseteq X \times X$ be a semi-independence alphabet, $SD = X \times X \setminus SI$, $SC = \{ab \Rightarrow ba \mid (a,b) \in SI\}$. Then the semi-commutation system SC is not confluent if and only if the semi-dependence alphabet (X, SD) contains

[1] For any relation $\mathscr{R} \subseteq \mathscr{S} \times \mathscr{S}$ of a set $\mathscr{S}$ the relation $\mathscr{R}^{-1}$ is defined by $\mathscr{R}^{-1} = \{(y, x) \mid (x, y) \in \mathscr{R}\}$.

a directed cycle going through at least two directed arcs but without any undirected chord.

In what follows we shall consider the relative situation $p: M \to M'$, where $M = M(X, I)$, $M' = M(X, I')$ such that $I \subseteq I'$ and p denotes the canonical projection. It is clear that p extends in a unique way to a surjective ring homomorphism

$$p: Z\langle\langle M\rangle\rangle \to Z\langle\langle M'\rangle\rangle, \quad \sum_{t \in M} f(t)t \mapsto \sum_{t' \in M'} \left(\sum_{p(t)=t'} f(t) \right) t'.$$

The notion of unambiguous lifting is defined with respect to p.

Definition. Let $\mu_{M'} \in Z\langle\langle M'\rangle\rangle$ be the Möbius function of M' and $\mu \in Z\langle\langle M\rangle\rangle$ be a formal power series over M. We call μ an *unambiguous lifting* of $\mu_{M'}$ if the following two conditions are satisfied:

(i) The function μ is the formal inverse of a characteristic series over a (rational) cross section of M' in M.

(ii) The support of μ maps bijectively onto $supp(\mu_{M'})$.

Remark 2.3. Note that in the special case where $p: M \to M'$ is the identity, we reobtain the original definition of the Möbius function by $\mu_{M'} = (\sum_{t \in M'} t)^{-1}$.

Directly from the definition above we can state two facts. The first one is immediate.

Proposition 2.4. *Let $I \subseteq I' \subseteq I'' \subseteq X \times X$ be independence relations and $M = M(X, I)$, $M' = M(X, I')$, $M'' = M(X, I'')$ the corresponding trace monoids. Let $\mu \in Z\langle\langle M\rangle\rangle$ be an unambiguous lifting of the Möbius function $\mu_{M''} \in Z\langle\langle M''\rangle\rangle$. Then the image of μ in $Z\langle\langle M'\rangle\rangle$ is an unambiguous lifting of $\mu_{M''}$ to the monoid M'.*

Proposition 2.5 has also a direct verification. However, later at the end of Section 5 we will provide a simpler proof based on confluent semi-commutations.

Proposition 2.5. *Let $M_i = M(X_i, I_i)$ and $M_i' = M(X_i, I_i')$ be trace monoids such that $I_i \subseteq I_i'$ and let $\mu_i \in Z\langle\langle M_i\rangle\rangle$ be unambiguous liftings of the Möbius function $\mu_{M_i'} \in Z\langle\langle M_i'\rangle\rangle$ for $i = 1, 2$. Denote by $M = M_1 * M_2 = M(X_1 \dot\cup X_2, I_1 \dot\cup I_2)$ the free product of M_1 and M_2. Then the Möbius functions of the direct product $M_1' \times M_2'$ and of the free product $M_1' * M_2'$ have both unambiguous liftings to $Z\langle\langle M\rangle\rangle$. More precisely,*

$$\mu_1 \cdot \mu_2 \in Z\langle\langle M\rangle\rangle$$

is an unambiguous lifting of the Möbius function $\mu_{M_1' \times M_2'}$ of the direct product $M'_1 \times M'_2$ and

$$\mu_1 + \mu_2 - 1 \in Z\langle\langle M\rangle\rangle$$

is an unambiguous lifting of the Möbius function $\mu_{M_1' * M_2'}$ of the free product $M'_1 * M'_2$.

3. From Möbius functions to semi-commutations

Our first result states that an unambiguous lifting yields in a natural way a confluent semi-commutation system.

Theorem 3.1. *Let (X,I) and (X,I') be dependence alphabets with $I \subseteq I'$ and let $\mu \in Z\langle\langle M\rangle\rangle$ be an unambiguous lifting of the Möbius function $\mu_{M'} \in Z\langle\langle M'\rangle\rangle$.*

Then $SC(\mu) = \{ab \Rightarrow ba \mid \mu(ab) = 1\}$ is a confluent semi-commutation system.

Proof. Let $\zeta = \mu^{-1} \in Z\langle\langle M\rangle\rangle$ be the formal inverse of μ. The key to the proof is the following sufficient condition such that $\zeta(x_1 \ldots x_m) = 1$.

Lemma 3.2. *Let $x_1, \ldots, x_m \in X$, $m \geqslant 0$ such that $\mu(x_{i-1}x_i) = 0$ for all $1 \leqslant i \leqslant m$. Then it holds that $\zeta(x_1 \ldots x_m) = 1$.*

Proof. First, observe that $\mu(x_{i-1}x_i) = 0$ implies that x_{i-1} and x_i are dependent, i.e., $(x_{i-1}, x_i) \in X \times X \setminus I$. Hence, for $m \geqslant 0$ the trace $x_1 \ldots x_m \in M$ is given by the graph $x_1 \to \cdots \to x_m$.

In order to prove the lemma by induction we show a slightly stronger result:

$$\zeta(x_1 \ldots x_m) = 1$$

$$\mu(x_1 \ldots x_m) = \begin{cases} 1 & \text{if } m = 0, \\ -1 & \text{if } m = 1, \\ 0 & \text{otherwise; i.e., } m \geqslant 2. \end{cases}$$

The formulae are obvious for $m = 0, 1$. For $m \geqslant 2$, the formula

$$-\zeta(x_1 \ldots x_m) = \mu(x_1)\,\zeta(x_2 \ldots x_m) + \cdots + \mu(x_1 \ldots x_m)$$

yields by induction on m the expression

$$1 = \zeta(x_1 \ldots x_m) + \mu(x_1 \ldots x_m)$$

Assume that we would have $\zeta(x_1 \ldots x_m) \neq 1$. Since μ is an unambiguous lifting this implies $\zeta(x_1 \ldots x_m) = 0$ and $\mu(x_1 \ldots x_m) = 1$. In particular, $(x_i, x_j) \in I'$ for $1 \leqslant i \neq j \leqslant m$ and, hence, $\mu(x_2 x_1) = 1$ since $\mu(x_1 x_2) = 0$. Now, consider the trace

$$x_2 \to x_1 \to x_2 \to \cdots \to x_m.$$

We have

$$0 \geqslant -\zeta(x_2 x_1 x_2 \ldots x_m) = \mu(x_2)\,\zeta(x_1 x_2 \ldots x_m) + \mu(x_2 x_1)\,\zeta(x_2 \ldots x_m) = 0 + 1 = 1.$$

This is a contradiction; hence, $\zeta(x_1 \ldots x_m) = 1$ and $\mu(x_1 \ldots x_m) = 0$ by the expression above. $\square$

$$t = \begin{tikzcd} x_1 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & x_m \\ y_n & \cdots \arrow[l] & y_1 \arrow[l] \arrow[u, dotted] \end{tikzcd}$$

Fig. 1.

Proof of Theorem 3.1 (*conclusion*). Assume now that $SC(\mu)$ would not be confluent and let $S(\mu)=\{ab\Rightarrow ba \mid \mu(ba)=1,\ \mu(ba)=0\}$. Then $S(\mu)$ is not locally confluent and by [8, Theorem 2.1] (or by Remark 2.2) we find two different rules $x_1 y_n \Rightarrow y_n x_1$, $y_1 x_m \Rightarrow x_m y_1 \in S$ and a trace t, whose Hasse diagram is shown in Fig. 1.

Note that we have $(m, n) \neq (1,1)$. Here the arcs from x_1 to y_n and from y_1 to x_m are drawn dotted in order to indicate that $x_1 y_n$ and $y_1 x_m$ are left-hand sides of the system $S(\mu)$.

Thus, we have

$$t = y_1 \ldots y_{n-1} x_1 y_n x_2 \ldots x_m = x_1 \ldots x_{m-1} y_1 x_m y_2 \ldots y_n$$

and we may apply a rule either to the dotted arc $x_1 y_n$ or to $y_1 x_m$.

We obtain

$$t_1 \underset{S(\mu)}{\Leftarrow} t \underset{S(\mu)}{\Rightarrow} t_2$$

such that

$$t_1 = y_1 \rightarrow \cdots \rightarrow y_n \rightarrow x_1 \rightarrow \cdots \rightarrow x_m,\ t_2 = x_1 \rightarrow \cdots \rightarrow x_m \rightarrow y_1 \rightarrow \cdots \rightarrow y_n \in Irr(S(\mu))$$

Clearly, $t_1 \neq t_2$ in M and Lemma 3.2 tells us that $\zeta(t_1) = \zeta(t_2) = 1$. However, since t_1 and t_2 are congruent modulo $S(\mu)$, they have the same image in M'. Thus, ζ has no cross section and this is a contradiction to the assumption that μ is an unambiguous lifting. $\square$

Remark 3.3. The proof above gives us no explicit description about the function $\zeta = \mu^{-1}$. In fact, it will follow only later from Theorem 5.1 that $\zeta = \sum_{t \in Irr(S(\mu))} t$ is the characteristic series of the irreducible traces with respect to the complete system $S(\mu) = \{ab \Rightarrow ba \mid \mu(ab) = 1,\ \mu(ba) = 0\}$. This is stated in Corollary 5.2. See also the concluding remarks below.

4. From semi-commutations to Möbius functions

The other way round, if $SC \subseteq X \times X$ is a confluent semi-commutation system then we show in this section that this yields in a canonical way an unambiguous lifting.

Definition. Let $SI \subseteq X \times X$ be a semi-independence alphabet with associated semi-commutation system $SC = \{ab \Rightarrow ba \mid (a,b) \in SI\}$. As usual, let $I = SI \cap SI^{-1}$, $I' = SI \cup SI^{-1}$, $M = M(X, I)$, $M' = (X, I')$ and $S = \{ab \Rightarrow ba \mid (a,b) \in SI, (b,a) \notin SI\}$. Then define the function $\mu(SC)$ by the formal inverse over the irreducible traces:

$$\mu(SC) = \left(\sum_{t \in Irr(S)} t \right)^{-1} \in Z\langle\langle M \rangle\rangle.$$

Theorem 4.1. *Let SC be a confluent semi-commutation system. Then the formal power series $\mu(SC)$ is an unambiguous lifting of the Möbius function $\mu_{M'}$.*

We will obtain the proof of Theorem 4.1 in several steps. First, let us give an explicit description of the function $\mu(SC)$. For this we use the reverse, $rev(t)$, of a trace t which is the trace t read from right to left; hence, the trace which is obtained from t by reversing all arcs in the dependence graph of the trace. If a trace t is given as a congruence class $t = [a_1 \ldots a_m]$ $m \geq 0$, $a_i \in X$ for $1 \leq i \leq m$ then we have $rev(t) = [a_m \ldots a_1]$. If SC is a confluent semi-commutation system then we may represent each clique $F' \in \mathscr{F}' = \mathscr{F}(X, I')$ by the reverse of the irreducible normal form with respect to S in $M(X, I)$. For $F' \in \mathscr{F}'$ we denote by $[\hat{F}'] \in Irr(S)$ the irreducible trace such that $p([\hat{F}']) = [F']$. In these notations we can state the more precise formula.

Theorem 4.2. *Let SC be confluent. Then we have $\mu(SC) = \sum_{F' \in \mathscr{F}'} (-1)^{|F'|} rev([\hat{F}']) \in Z\langle\langle M \rangle\rangle$.*

Of course, Theorem 4.2 implies Theorem 4.1. Note that for $ab \Rightarrow ba \in S$ we have $\{a, b\} \in \mathscr{F}'$, $[\{a \hat{,} b\}] = ba \in Irr(S)$, and $rev([\{a \hat{,} b\}]) = ab$. Hence, $\mu(ab) = 1$, $\mu(ba) = 0$ in this case. For dependence graphs of traces we distinguish (with respect to S) between two types of arcs. An arc from a vertex with label a to a vertex with label b is called *hard* (with respect to S) if $ab \in Irr(S)$. It is drawn as $a \rightarrow b$. If ab is the left-hand side of a rule of S then the arc from a to b is called *soft* (with respect to S) and it is drawn dotted $a \cdots\cdots\blacktriangleright b$. Note that a trace is reducible if and only if its Hasse diagram contains some soft arc.

In what follows we call a trace *soft* (with respect to S) if its dependence graph contains soft arcs only. Of course, soft traces yield cliques in M'. Lemma 4.3 implies that every clique can be represented by a soft trace.

Lemma 4.3. *Let SC be confluent. Then a trace $s \in M$ is soft if and only if $rev(s)$ is the irreducible normal form of a clique in $\mathscr{F}'$.*

Proof. Let $s \in M$ be soft, then the image of $s \in M'$ is obviously a clique. Since every soft arc $a \cdots\cdots\blacktriangleright b$ becomes a hard arc $b \rightarrow a$ in $rev(s)$, the trace $rev(s)$ contains hard arcs, only. In particular, $rev(s) \in Irr(S)$. For the other direction let $F' \in \mathscr{F}'$ be a clique and $s = [\hat{F}'] \in M$ be its irreducible normal form. We have to show that $rev(s)$ is soft, or,

$$\begin{tikzcd} x_2 \arrow[r] & \cdots \arrow[r] & x_{m-1} \arrow[d] \\ x_1 \arrow[u] \arrow[rr, dotted] & & x_m \end{tikzcd}$$

Fig. 2.

$$\begin{tikzcd} x_2 \arrow[r] \arrow[d, dotted] & \cdots \arrow[r] & x_{m-1} \arrow[d] \\ x_1 & & x_m \arrow[ll] \end{tikzcd}$$

Fig. 3.

equivalently, that the dependence graph of s does not contain any soft arc. Assume the contrary that s has some soft arc. Since s is irreducible the Hasse diagram of s has no soft arcs. Hence, we find a minimal subgraph in s as shown in Fig. 2, with $m \geqslant 3$, $x_{i-1} \to x_i$ is a hard arc for $1 < i \leqslant m$ and $x_1 \cdots\cdots\blacktriangleright x_m$ is soft. Clearly, $t' = x_1 \ldots x_m \in Irr(S)$ and by definition of s the image of t' is a clique in $\mathscr{F}'$. Now consider $t'' = x_2 x_3 \ldots x_m x_1$; this trace looks as shown in Fig. 3.

Hence, $t'' \in Irr(S)$, too. Since $t' \neq t''$, but t' and t'' have the same image in M' this is a contradiction to the confluence of SC. $\square$

By Lemma 4.3, Theorems 4.1 and 4.2 follow immediately from the following fact.

Theorem 4.4. *Let SC be confluent. Then we have $\mu(SC) = \sum_{s\ soft} (-1)^{|s|} s \in Z\langle\langle M \rangle\rangle$.*

Recall that by the definition of $\mu(SC)$ we have to show that $(\sum_{t \in Irr(S)} t)^{-1} = \sum_{s\ soft} (-1)^{|s|} s \in Z\langle\langle M \rangle\rangle$. The proof of Theorem 4.4 is quite involved. We start with the following lemma.

Lemma 4.5. *Every trace $t \in M$ has a maximal soft prefix and a maximal irreducible suffix (with respect to S).*

Proof. Let $x, y, z, w \in M$ be traces such that $xy = zw$. Then by Levi's Lemma, see Cori and Perrin [5], we find $r, u, v, s \in M$ such that $x = ru$, $y = vs$, $z = rv$, $w = us$ and $alph(u) \times alph(v) \subseteq I$. (This factorization can be seen in Fig. 4.)

We have to show that

(a) if x, z are soft then $ruv = rvu$ is soft, too,

(b) if y, w are irreducible then $uvs = vus$ is irreducible, too.

By Levi's Lemma we know that in the dependence graph of $xy = zw$ there is no arc from $u = x \cap w$ to $v = y \cap z$ nor vice versa. Hence, we obtain (a), since the trace ruv

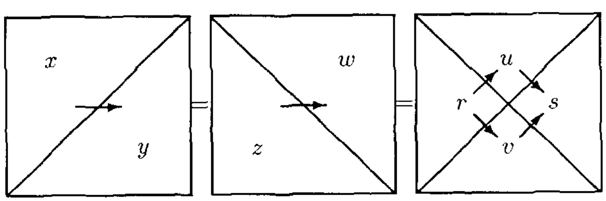

Fig. 4. The factorization by Levi's Lemma.

contains soft arcs only; we obtain (b), since the Hasse diagram of uvs has hard arcs only. □

Proof of Theorem 4.4. Let $\zeta=\sum_{t\in Irr(S)} t$ and $\mu=\sum_{s\ soft}(-1)^{|s|}s$. We have to show that $(\mu\zeta)(1)=1$ and $\mu\zeta(t)=0$ for $t\neq 1$. The equation $(\mu\zeta)(1)=1$ is clear. Thus, consider $t\neq 1$. Let $x\subseteq t$ be the maximal soft prefix of t and $w\subseteq t$ be the maximal irreducible suffix of t. By Lemma 4.5 x and w are well-defined and since $t\neq 1$ we have $x\neq 1$ and $w\neq 1$. By Levi's Lemma we find $r,u,v,s\in M$ such that $t=ruvs$, $x=ru$, $w=us$ and $alph(u)\times alph(v)\subseteq I$. Assume first that we have $v\neq 1$. Then, there is no factorization $t=pq$ such that p is soft, and q is irreducible. Hence, $(\mu\zeta)(t)=\sum_{pq=t}\mu(p)\zeta(q)=0$ in this case. Therefore, we may assume that $v=1$ and $t=rus$. Next observe that there is a one-to-one correspondence between factorizations of u and pairs (p,q) such that $pq=t$, p is soft and q is irreducible. This means that there is a bijection between the set $\{(u_1,u_2)\in M\times M \mid u_1u_2=u\}$ and the set $\{(p,q)\in M \mid pq=t,\ p \text{ is soft, } q \text{ is irreducible}\}$. Since the bijection is induced by $(u_1,u_2)\mapsto(ru_1,u_2s)$, we can compute

$$(\mu\zeta)(t)=\sum_{pq=t}\mu(p)q(t)=\sum_{u_1u_2=u}\mu(ru_1)\zeta(u_2s).$$

The trace ru_1 is soft and the trace u_2s is irreducible for all factorizations $u_1u_2=u$. Therefore, $\mu(ru_1)=(-1)^{|r|}(-1)^{|u_1|}$ and $\zeta(u_2s)=1$ for $u_1u_2=u$. Thus, we have to show that

$$\sum_{u_1u_2=u}(-1)^{|u_1|}=0.$$

Recall that u is soft and irreducible at the same time. Hence, the dependence graph of u has no arcs at all and the trace u is a clique in M. This means that it can be identified with a subset $F\subseteq X$ such that $(a,b)\in I$ for all $a,b\in F$, $a\neq b$.

Now for $u\neq 1$ we can apply Foata's original trick and we obtain

$$\sum_{u_1u_2=u}(-1)^{|u_1|}=\sum_{F_1\subseteq F}(-1)^{|F_1|}=\sum_{k=0}^{n}\binom{n}{k}(-1)^k=(1-1)^n=0,$$

with $n=|F|$ and $F\neq\emptyset$.

Therefore, the proof of Theorem 4.4 is complete if $u\neq 1$. The crucial point, however, is to show that in the above situation $u=1$ is impossible. Thus, let $t\neq 1$ and assume we would have a factorization $t=xw$ such that x is the maximal soft prefix and w is the maximal irreducible suffix.

$$t=\boxed{\begin{array}{c|c} x & w \\ (soft) & (irreducible)\end{array}}\,.$$

Next, consider $t\overset{*}{\underset{S}{\Rightarrow}}\hat{t}\in Irr(S)$. The dependence graph of $\hat{t}$ has the same labelled vertex set as t. We obtain $\hat{t}$ from t by turning soft arcs in the Hasse diagram of t into hard arcs of the opposite direction. No hard arc of t vanishes in this procedure. Since the system

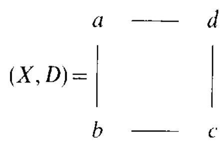

Fig. 5.

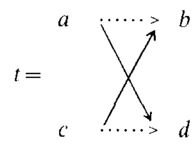

Fig. 6.

S is confluent, we can start to compute $\hat{t}$ by turning any soft arc in the Hasse diagram of t. The confluence of S is used in the sense that the order in which we turn soft arcs is irrelevant. Let us identify vertices of x and vertices of w with the corresponding vertices of $\hat{t}$. Let $b \in w$ be minimal in $w \subseteq t$. Since xb is not a soft trace there exists some $a \in x$ and a hard arc $a \rightarrow b$ in t. Since hard arcs do not vanish in $\hat{t}$ it follows that no point of w can be minimal in $\hat{t}$. Now, let $a \in x$ be maximal in the soft trace $x \subseteq t$. Since $aw \notin Irr(S)$ there is some $b \in w$ and a soft arc $a \cdots\cdots> b$ in the Hasse diagram of t. It follows that from every vertex of x there is a soft path in the Hasse diagram to some vertex of w. By the confluence of S we may start to compute $\hat{t}$ by turning the first soft arc of such a path. This implies that no vertex of x will be minimal in $\hat{t}$. Summarizing, we have shown that no vertex at all of t will be minimal in $\hat{t}$. This is impossible. $\square$

Example. The proof above used essentially the following fact. If the system S is confluent and $t \in M$ is a trace with a maximal soft prefix x and a maximal irreducible suffix w, then we never have $t = xw$. If the system S is not confluent then this assertion becomes false. To see this let $S = \{ab \Rightarrow ba,\ cd \Rightarrow dc\}$. We view S as a system over the trace monoid with the following dependence alphabet (Fig. 5). Then S is not confluent and the shortest trace where nonconfluence can be established is Fig. 6. The maximal soft prefix of t is the clique $x = ac$ and the maximal irreducible suffix is the clique $w = bd$. However, we have $t = xw$.

Remark 4.6. The example above can be easily generalized to the trace t used in the proof of Theorem 3.1. Together with [8, 21] we can say that a semi-commutation system is not confluent if and only if there exists a trace which is the product of its maximal soft prefix and its maximal irreducible suffix. This yields another characterization of confluence.

Corollary 4.7. *Let $M = M(X, I)$ and $M' = M(X, I')$ be trace monoids such that $I \subseteq I'$. Then there exists an unambiguous lifting of the Möbius function $\mu_{M'} \in Z\langle\langle M' \rangle\rangle$ to the ring $Z\langle\langle M \rangle\rangle$ if and only if there is a confluent semi-commutation system $SC = \{ab \Rightarrow ba \mid (a, b) \in SI\}$ such that $I = SI \cap SI^{-1}$ and $I' = SI \cup SI^{-1}$.*

Proof. Combine Theorems 3.1 and 4.1. $\square$

Example. Let $X = \{a, b, c, d, e, f, g, h\}$. Define two dependence relations: (X, D') is given by the cycle (a, c, b, d, g, e, h, f), and $(X, D) = (X, D') \cup \{ag, bh\}$. If we draw (X, D') by

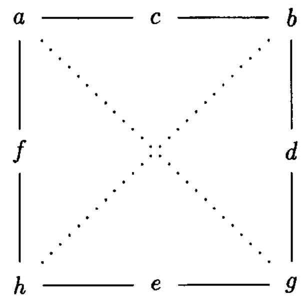

Fig. 7. (X, D') = straight edges, (X, D) = straight and dot edges.

straight (hard) edges and the difference of (X, D) and (X, D') by dotted (soft) edges, then we have Fig. 7.

Now, a simple inspection of this picture shows that for every orientation of the soft edges we find a trace (of length six) of the same type which appeared in the proof of Theorem 3.1. This shows that for no orientation of the equations $ag = ga$, $bh = hb$ we will obtain a confluent semi-commutation system with respect to (X, D) and (X, D'). (Compare also with Remark 2.2.) Hence, by Corollary 4.7 there is no unambiguous lifting of the Möbius function in this situation.

Remark 4.8. The example above settles also a conjecture of [12]: Following the terminology of this conjecture, the complement of the graph (X, D') is the commutation graph (A, θ). We may assume that the ordering on vertices is lexicographically $a < b < c < d < e < f < g < h$. Now, the construction of (A, ϕ) in [12, Conjecture 5.2] yields that $\{a, g\} \notin \phi$ and $\{b, h\} \notin \phi$. The reason for this is that we have $\{a, h\} \in \theta$. Conjecture 5.2 of [12] states that in such a situation it should be possible to lift the Möbius function unambiguously from (A, θ) to (A, ϕ). Then, using Proposition 2.4, we could also lift in the example above. But we have just seen that this is impossible and, hence, Conjecture 5.2 of [12] cannot be true. (Contrary to a statement in a preliminary version of the present paper.)

5. The bijection

It turns out that the situation is in fact much nicer than described in Corollary 4.7, which is purely existential. In some sense confluent semi-commutation systems and unambiguous liftings of Möbius functions are the same. This is expressed in the following theorem which can be viewed as the main result of the paper. Unfortunately, its proof is very technical and needs some tedious computations, in contrast to the simplicity of the statement itself.

Theorem 5.1. *There is a canonical one-to-one correspondence between confluent semi-commutation systems and unambiguous liftings of Möbius functions.*

Proof. In the parts above we have constructed a mapping $\mu \mapsto SC(\mu)$ from unambiguous liftings of Möbius functions to confluent semi-commutation systems, and a mapping $SC \mapsto \mu(SC)$ in the inverse direction. The equation $SC = SC(\mu(SC))$ is trivial. Hence, the proof of Theorem 5.1 reduces to show that we have $\mu(SC(\mu)) = \mu$ for all unambiguous liftings of Möbius functions. We do this by contradiction. Let μ be any unambiguous lifting, $SC = SC(\mu) = \{ab \Rightarrow ba \mid \mu(ab) = 1\}$, $S = S(\mu) = \{ab \Rightarrow ba \mid \mu(ab) = 1, \mu(ba) = 0\}$, and $\mu' = \mu(SC)$. Define $\zeta = \mu^{-1}$ and $\zeta' = \mu'^{-1}$ in $Z\langle\langle M \rangle\rangle$. Assume that we would have $\mu' \neq \mu$, then we have $\mu \neq \mu' = \sum_{s\, soft} (-1)^{|s|} s$ and $\zeta \neq \zeta' = \sum_{t \in Irr(S)} t$. Here the notion of soft trace is defined as in the preceding section with respect to the confluent system S. Note that μ' and ζ' have explicit descriptions by the formulae above whereas for the moment not much is known about μ and ζ, which are functions we are interested in.

Consider a trace $t \in M$ of minimal length such that $\zeta(t) \neq \zeta'(t)$ or $\mu(t) \neq \mu(t')$. It is clear that the length, $|t|$, of t is at least three.

Since

$$-\zeta(t) = \sum_{\substack{pq = t \\ p \neq 1}} \mu(p)\zeta(q),$$

$$-\zeta'(t) = \sum_{\substack{pq = t \\ p \neq 1}} \mu'(p)\zeta'(q),$$

we obtain by minimality of the length of t the equation

$$\zeta(t) - \zeta'(t) = \mu'(t) - \mu(t).$$

Since μ and μ' are unambiguous liftings we may assume that $\zeta'(t) = 1$ and $\zeta(t) = 0$. Indeed, if we should have $\zeta'(t) = 0$ and $\zeta(t) = 1$ then replace t by the trace t' of same length such that $\zeta'(t') = 1$ and which has the same image in M' as t. Thus, $\zeta'(t) = 1$ and $\zeta(t) = 0$ holds without restriction. It follows that $\mu(t) = 1$, $\mu'(t) = 0$ if $|t|$ is even and $\mu(t) = 0$, $\mu'(t) = -1$ if $|t|$ is odd.

In any case, since $\mu(t) \neq 0$ or $\mu'(t) \neq 0$, the image of t in M' is a clique. But t itself cannot be a clique of M since otherwise $\mu(t) \neq \mu'(t)$ would be impossible. By the definition of ζ' we know that t is irreducible with respect to S. Now, if we would have $\mu'(t) \neq 0$ then t would also be soft. But the only traces which are irreducible and soft with respect to S are the cliques. Since t is not a clique we deduce that we have $\mu(t) = 1$, $\mu'(t) = 0$ and the length $|t|$ is even. Since t is irreducible and not soft we see that t contains some hard arc $a \to b$, where $a \in t$ is minimal. Then the trace $bt \in M$ is reducible and we can compute:

$$\begin{aligned}
0 &\leq \zeta(bt) \\
&= \zeta(bt) - \zeta'(bt) \\
&= \sum_{\substack{pq = bt \\ p \neq 1}} (\mu'(p)\zeta'(q) - \mu(p)\zeta(q)) \\
&= -1 + \sum_{\substack{pq = bt \\ p \neq 1, b}} (\mu'(p)\zeta'(q) - \mu(p)\zeta(q)).
\end{aligned}$$

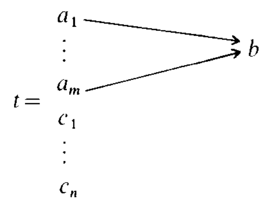

Fig. 8.

Now, consider any pair (p, q) such that $pq = bt$, $p \neq 1, b$. If the length $|p| = 1$ then $p \neq b$ and the image of q is not a clique in M'. But then $\zeta(q) = \zeta'(q)$ since $|q| = |t|$. (Recall from the discussion above that a trace yields a clique in M' if it is of minimal length where ζ and ζ' have different values.) If $1 < |p| < |t|$ then we have $|p| < |t|$ and $|q| < |t|$. Hence, $\mu'(p)\zeta'(p) = \mu(p)\zeta(p)$ for all $pq = bt$, $p \neq 1, b, |p| < |t|$. The same holds for $pq = bt$ if $|p| = |t|$ and $q \neq b$ since this implies $\mu(p) = \mu'(p) = 0$.

Thus, if there is no factorization $pb = bt$ then $\sum_{pq=bt,\, p\neq 1,\, b}(\mu'(p)\zeta'(q) - \mu(p)\zeta(q)) = 0$ and we have a contradiction $0 \leqslant -1$. Therefore, $pb = bt$ is possible for some $p \in M$ and we must have $\mu'(p) = 1$, $\mu(p) = 0$, $\zeta(bt) = 0$ since the length $|p| = |t|$ is even.

Next observe that $pb = bt$ implies that the image of p and of t in M' is the same clique. Since $\mu'(rev(t)) = 1$ we have $p = rev(t)$, i.e., $rev(t)b = bt$. Noting that $\zeta(rev(t)) = 1$ follows by $\zeta'(rev(t)) - \zeta(rev(t)) = \mu(rev(t)) - \mu'(rev(t))$, we obtain the following values:

$$\zeta'(t) = 1, \qquad \mu'(t) = 0, \qquad \zeta'(rev(t)) = 0, \qquad \mu'(rev(t)) = 1,$$

$$\zeta(t) = 0, \qquad \mu(t) = 1, \qquad \zeta(rev(t)) = 1, \qquad \mu(rev(t)) = 0.$$

The equation $rev(t)b = bt$ yields that t looks as shown in Fig. 8, with $a = a_1$, $m + n \geqslant 3$, odd, i.e., $alph(t) = \{a_1, \ldots, a_m, b, c_1, \ldots, c_n\}$, $m \geqslant 1$, $m + n \geqslant 3$, odd, and the arcs of t are from a_i to b for $1 \leqslant i \leqslant m$.

In particular, t has exactly one element b, which is maximal without being minimal. Next, we start to compute $\zeta(ta)$. This is a symmetric situation and the calculation is left to the reader. It yields that t has exactly one element which is minimal without being maximal. Thus, we have $m = 1$, which gives Fig. 9 for the trace t. Since $|t| \geqslant 3$ and $|t|$ is even, we have $n \geqslant 2$. The final computation is done on the trace shown in Fig. 10.

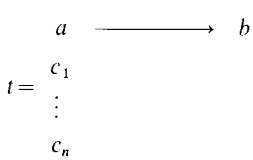

Fig. 9.

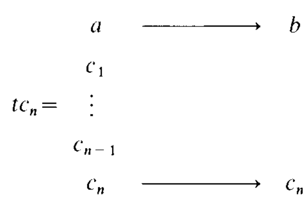

Fig. 10.

We will compute $\zeta(tc_n)-\zeta'(tc_n)$. For this, consider $pq=tc_n$ with $1<|p|<|t|$, then $|q|<|t|$ and $\mu'(p)\zeta'(q)=\mu(p)\zeta(q)$. If $|p|=1$ and $p\neq c_n$ then q is not a clique in M' and, hence, $\mu'(p)\zeta'(q)=\mu(p)\zeta(q)$, too. If $|p|\geqslant|t|$ and $q\neq c_n$ then $\mu'(p)\zeta'(q)=\mu(p)\zeta(q)$ since $\mu'(p)=\mu(p)=0$.

It follows that

$$\begin{aligned}\zeta(tc_n)-\zeta'(tc_n)&=\sum_{\substack{pq=tc_n\\ p\neq 1}}(\mu'(p)\zeta'(q)-\mu(p)\zeta(q))\\ &=(\mu'(c_n)\zeta'(t)-\mu(c_n)\zeta(t)+\mu'(t)\zeta'(c_n)-\mu(t)\zeta(c_n)\\ &=(-1)-0+0-(+1)\\ &=-2.\end{aligned}$$

This is a contradiction. □

Corollary 5.2. *Let $\mu\in Z\langle\langle M\rangle\rangle$ be an unambiguous lifting of a Möbius function $\mu_{M'}$ and $S=S(\mu)=\{ab\Rightarrow ba\mid\mu(ab)=1,\mu(ba)=0\}\subseteq M\times M$ the associated complete system. Then we have the identity $\mu=(\sum_{t\in Irr(S)}t)^{-1}$.*

Proof. By Theorem 5.1 we have $\mu(SC)=\mu$ and $\mu(SC)=(\sum_{t\in Irr(S)}t)^{-1}$ by the definition in Theorem 4.4. □

We are now also able to give the simple proof announced above.

Proof of Proposition 2.5. Without restriction we have $X_1\cap X_2=\emptyset$. By Theorem 5.1 we can assume that $\mu_i=\mu(SC_i)$ for confluent semi-commutation systems $SC_i\subseteq X_iX_i\times X_iX_i$, $i=1,2$. It is clear that $SC=SC_1\cup SC_2\cup\{ab\Rightarrow ba\mid a\in X_1,\ b\in X_2\}$ is a confluent semi-commutation system for the direct product $M_1'\times M_2'$. The formula $\mu(SC)=\mu_1\mu_2$ follows from Theorem 4.4. For the free product $M_1 * M_2$ it is enough to consider the confluent system $SC_1\cup SC_2$. □

Let us now consider two special cases of the relative situation $p: M(X,I)\to M(X,I')$. The first one is $I=I'$. The interesting fact here is that we reobtain the original formula of Foata for the Möbius function; without that we have ever used this description.

Indeed, if $p=id_M$ then $SC=C$ and a symmetric system is obviously confluent. The formula $(\sum_{t\in Irr(S)}t)^{-1}=\sum_{s\ soft}(-1)^{|s|}s$ from Theorem 4.4 then becomes $(\sum_{t\in M}t)^{-1}=\sum_{F\in\mathcal{F}}(-1^{|F|}[F])$. This is clear since in this situation a trace is soft if and only if it is a clique of the independence alphabet.

The second special case is $I=\emptyset$. This has been the starting point of [6] and the results above are a generalization of [6, Theorem 12′]. In particular, they also generalize the results from [16, 19].

In the case $I=\emptyset$ all related decidability questions refer to comparability graphs can be answered in polynomial time; see [9]. In the relative situation these questions are much more difficult. Using Theorem 5.1, we can apply the complexity results of [8] to state the completeness results of Section 6.

6. Complexity results

The proofs of this section are reductions to [8] where the results are shown for the analogous problems for confluent semi-commutation systems.

Theorem 6.1. *The following problem is* Co-NP-*complete. Given a semi-independence alphabet $SI \subseteq X \times X$. Does there exist an unambiguous lifting μ of the Möbius function from $M(X, SI \cup SI^{-1})$ to the ring of formal power series over $M(X, SI \cap SI^{-1})$ such that $\mu(ab) = 1$ if and only if $(a, b) \in SI$?*

Proof. The same question for semi-commutation systems is whether $SC = \{ab \Rightarrow ba \mid (a, b) \in SI\}$ is confluent. This question is Co-NP-complete by [8, Theorem 3.3].

Remark 6.2. The problem above is Co-NP-complete even in the restricted case where $SI \setminus (SI \cap SI^{-1})$ contains at most two pairs $(a, b), (c, d)$. Thus, we have the following situation. If $SI \setminus (SI \cap SI^{-1})$ contains only one pair (a, b) then there exists a unique unambiguous lifting. This is trivial. Soon as there are at least two pairs the problem becomes Co-NP-complete. In terms of semi-commutation this means that for the fixed system of two rules $S = \{ab \Rightarrow ba, cd \Rightarrow dc)$ it is Co-NP-complete to decide whether on input $M = M(X, I)$ with $a, b, c, d \in X$ the system $S \subseteq M \times M$ is complete.

The complexity question which may be the most natural in our context goes even (as far as we believe) beyond NP and Co-NP. Let Σ_2^P denote the second level of the polynomial hierarchy, i.e., Σ_2^P is the class of languages recognized by an NP-machine having access to some NP-oracle.

Theorem 6.3. *The following problem is Σ_2^P-complete. Given trace monoids $M = M(X, I)$, $M' = M(X, I')$ such that $I \subseteq I'$. Does there exist an unambiguous lifting of the Möbius function $\mu_{M'}$ to $Z\langle\langle M \rangle\rangle$?*

Proof. The same question for trace replacement systems is whether there exists a finite complete system $S \subseteq M \times M$ such that $M/S = M'$. (See Remark 2.1.) This question is Σ_2^P-complete by [8, Theorem 3.1]. □

If we bound the cardinality $\#(I' \setminus I)$ by some constant then (as we can imagine from Theorem 6.1) we are not better than Co-NP, again.

Theorem 6.4. *Let $k \geqslant 2$ be a fixed constant. Then the problem of Theorem* 6.3 *restricted to $\#(I' \setminus I) \leqslant 2k$ is* Co-NP-*complete.*

Proof. The same question for trace replacement systems (semi-commutation systems) bounds the number of rules (asymmetric rules) by k. The Co-NP-completeness of this is shown in [8, Theorem 3.2]. $\square$

On the other hand, if we have $I=\emptyset$ then we can decide all problems above in polynomial time. However, the complexity is not known if we bound the cardinality of I by some constant.

Open problem (Diekert et al. [8]). Let $k \geqslant 2$ be a fixed constant. What is the complexity of the problem of Theorem 6.3 restricted to the cases where $\# I \leqslant k$?

7. Concluding remarks

The origin of this work has been the question whether Möbius functions have unambiguous liftings to words. The answer of the author presented at ICALP '88, [6], showed the coincidence of such liftings and complete semi-Thue systems.

The generalization to the relative situation $p: M(X, I) \rightarrow M(X, I')$ was stated in [6] as an open problem. The assertions of Theorems 3.1, 4.4, and 5.1 were implicitly conjectured. But the solution in the case of liftings to words suggested to prove Theorem 3.1 via a direct verification of the formula $\mu^{-1} = \sum_{t \in Irr(S(\mu))} t$. This failed and only when the graph-theoretic characterization for the confluence of semi-commutations became clear, a different approach to Theorem 3.1 has been successful. (The formula mentioned above is now Corollary 5.2.) However, a direct proof of Corollary 5.2 would still be useful. Probably, it would lead to a new proof in terms of Möbius functions for the graph-theoretic characterization of confluence which is given in [8, 21]. Another point of interest is the homological interpretation given by Kobayashi [11]; see also [10]. It is very likely that this can also be extended to the general setting we have considered here. The interest in this is the direct relation to the techniques of Squier [22]. Squier proved the existence of finitely presented monoids with decidable word problem but without any complete presentation. We think that a relation of complete presentations and Möbius functions in a more general setting would give us more insight in both theories.

Acknowledgment

I am grateful to M. Ito, Y. Kobayashi, and R. König for various support of this work. I thank also the anonymous referee for pointing out several misprints.

References

[1] I.J. Aalbersberg and G. Rozenberg, Theory of traces, *Theoret. Comput. Sci.* **60** (1988) 1–82.

[2] P. Cartier and D. Foata, *Problèmes combinatoires de commutation et rèarrangements*, Lecture Notes in Mathematics, Vol. 85 (Springer, Berlin, 1969).

[3] C. Choffrut, Free partially commutative monoids, Tech. Report 86/20, LITP Université de Paris 7, 1986.

[4] M. Clerbout, Commutations partielles et familles de langages, Thése, Université de Lille, France, 1984.

[5] R. Cori and D. Perrin, Automates et commutations partielles, *RAIRO Inform. Théor. Appl.* **19** (1985) 21–32.

[6] V. Diekert, Transitive orientations, Möbius functions and complete semi-Thue systems for free partially commutative monoids, in: T. Lepistö and A. Salomaa, eds., *Proc. 15th Internat. Coll. on Automata Languages and Programming (ICALP '88)*, Tampere, Finland, 1988, Lecture Notes in Computer Science, Vol. 317 (Springer, Berlin, 1988) 176–187.

[7] V. Diekert, *Combinatorics on Traces*, Lecture Notes in Computer Science, Vol. 454 (Springer, Berlin, 1990).

[8] V. Diekert, E. Ochmanski and K. Reinhardt, On confluent semi-commutation systems – decidability and complexity results, in: J.L. Albert, B. Monien and M. Rodríguez Artalejo, eds., *Proc. 18th Internat. Coll. on Automata Languages and Programming (ICALP '91)*, Madrid, Spain, 1991, Lecture Notes in Computer Science, Vol. 510 (Springer, Berlin, 1991) 229–241; an extended version to appear in *Inform. and Comput.*

[9] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs* (Academic Press, New York, 1980).

[10] Y. Kobayashi, Complete rewriting systems and homology of monoid algebras, *J. Pure. Appl. Algebra* **65** (1990) 263–275.

[11] Y. Kobayashi, Partial commutation, homology and the Möbius inversion formula, in: M. Ito, ed., *Words, Languages and Combinatorics*, Kyoto, 1990 (World Scientific, Singapore, 1992) 288–298.

[12] R. König, Graphs and free partially commutative monoids, *Theoret. Comput. Sci.*, **78** (1991) 319–346.

[13] M. Latteux, Some results on semi-commutations, in: M. Ito, ed., *Words, Languages and Combinatorics*, Kyoto, 1990 (World Scientific, Singapore, 1992) 326–336; appeared also as Tech. Report I.T 201 Université de Lille, France.

[14] A. Mazurkiewicz, Concurrent program schemes and their interpretations, DAIMI Report PB 78, Aarhus University, Aarhus, 1977.

[15] A. Mazurkiewicz, Trace theory, in: W. Brauer, W. Reisig and G. Rozenberg, eds., *Petri Nets, Applications and Relationship to other Models of Concurrency*, Lecture Notes in Computer Science, Vol. 255 (Springer, Berlin, 1987) 279–324.

[16] Y. Métivier and E. Ochmanski, On lexicographic semi-commutations, *Inform. Process Lett.* **26** (1987) 55–59.

[17] M.H.A. Newman, On theories with a combinatorial definition of "equivalence", *Ann. of Math.* **43** (1942) 223–243.

[18] E. Ochmanski, Semi-commutations for place/transition systems, *Bull. EATCS* **38** (1989) 191–198.

[19] F. Otto, Finite canonical rewriting systems for congruences generated by concurrency relations, *Math. Systems Theory*, **20** (1987) 253–260.

[20] D. Perrin, Partial commutations, in: G. Ausiello, M. Dezani-Ciancaglini and S. Ronchi della Rocca, eds., *Proc. 16th Internat. Coll. on Automata, Languages and Programming (ICALP '89)*, Stresa, Italy, 1989, Lecture Notes in Computer Science, Vol. 372 (Springer, Berlin, 1989) 637–651.

[21] Y. Roos and P.A. Wacrenier, Composition of two semi commutations, in: A. Tarlecki, ed., *Proc. 16th Symp. on Mathematical Foundations of Computer Science (MFCS '91)*, Kazimierz Dolny, Poland, 1991, Lecture Notes in Computer Science, Vol. 520 (Springer, Berlin, 1991) 406–414; also available as Report I.T. 193, University of Lille, France.

[22] C. Squier, Word problems and a homological finiteness condition for monoids, *J. Pure and Appl. Algebra* **49** (1987) 201–218.